Explanatory Notes to Senior Secondary Mathematics Curriculum — Module 2 (Algebra and Calculus)

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Contents

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Foreword

The *Mathematics Curriculum and Assessment Guide (Secondary 4 – 6)* (2007) (abbreviated as "C&A Guide" in this booklet) has been prepared to support the new academic structure implemented in September 2009. The Senior Secondary Mathematics Curriculum consists of a Compulsory Part and an Extended Part. The Extended Part has two optional modules, namely Module 1 (Calculus and Statistics) and Module 2 (Algebra and Calculus).

In the C&A Guide, the Learning Objectives of Module 2 are grouped under different learning units in the form of a table. The notes in the "Remarks" column of the table in the C&A Guide provide supplementary information about the Learning Objectives. The explanatory notes in this booklet aim at further explicating:

1. the requirements of the Learning Objectives of Module 2;
2. the strategies suggested for the teaching of Module 2;
3. the connections and structures among different learning units of Module 2; and
4. the curriculum articulation between the Compulsory Part and Module 2.

The explanatory notes in this booklet together with the Remarks column and the suggested lesson time of each learning unit in the C&A Guide are to indicate the breadth and depth of treatment required. Teachers are advised to teach the contents of the Compulsory Part and Module 2 as a connected body of mathematical knowledge and develop in students the capability to use mathematics to solve problems, reason and communicate. Furthermore, it should be noted that the ordering of the Learning Units and Learning Objectives in the C&A Guide does not represent a prescribed sequence of learning and teaching. Teachers may arrange the learning content in any logical sequence which takes account of the needs of their students.

Comments and suggestions on this booklet are most welcomed. They should be sent to:

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Foundation Knowledge Area

The content of Foundation Knowledge Area comprises five Learning Units and is considered as the pre-requisite knowledge for Calculus Area and Algebra Area of Module 2. These Learning Units serve to bridge the gap between the Compulsory Part and Module 2. Therefore, it should be noted that complicated treatment of topics in this Area is not the objective of the Curriculum.

The Learning Unit “Surds” provides a necessary tool to help students manipulate limits and derivatives in Calculus Area. The Learning Unit “Binomial Theorem” forms the basis of the proofs of some rules in the Learning Unit “Differentiation”. Students should be able to prove propositions by applying mathematical induction. The Learning Unit “More about trigonometric functions” introduces the radian measure of angles, the six trigonometric functions and some trigonometric formulae commonly used in the learning of Calculus. Students should understand the importance of the radian measure in Calculus Area. The Learning Unit “Introduction to the number e” helps students understand that the natural logarithm is an important concept in mathematics and is crucial in differentiation as well as integration in Calculus.

As there is a strong connection between Foundation Knowledge Area and other Areas, teachers should arrange suitable teaching sequences to suit their students’ needs. For example, teachers can embed the Learning Unit “Surds” into Learning Unit 6 “Limits” when teaching differentiation from first principles to form a coherent set of learning contents.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
1 Surds	1.1 rationalise the denominators of expressions of the form $\dfrac{k}{\sqrt{a} \pm \sqrt{b}}$	1.5

Explanatory Notes:

The main focus of this Learning Unit is to rationalise the denominators of expressions of the form $\dfrac{k}{\sqrt{a} \pm \sqrt{b}}$. Teachers can point out that the expressions $\dfrac{k}{\sqrt{a} \pm \sqrt{b}}$ and $m(\sqrt{a} \mp \sqrt{b})$ are in fact of the same kind since one can be transformed into another by rationalisation. This technique helps students manipulate limits in Learning Unit 6. Students may apply this technique to rationalise the denominator as well as the numerator in finding some limits such as $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$. Teachers may skip this Learning Unit until the teaching of limits.

Teachers should be reminded that students may not have the concept of rationalisation at KS3. Only those who have studied the Learning Objective "Rationalisation of the denominator in the form of $\sqrt{a}$" of the Non-foundation Part in Learning Unit "Rational and Irrational Numbers" at KS3 may have come across this concept.

The identity $a^2 - b^2 \equiv (a-b)(a+b)$ should be reviewed as a pre-requisite knowledge for this Learning Unit.

Teachers should be aware that the rationalisation of denominators involving three or more unlike surds such as $\dfrac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}$ is not required.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
2. Mathematical induction	2.1 understand the principle of mathematical induction	5

Explanatory Notes:

Mathematical induction is an important tool in proving mathematical propositions. In Module 2, students are required to use it to prove propositions related to the summation of a finite sequence and divisibility.

In the C&A Guide, the term “understand” usually implies a more demanding learning objective than the term “recognise” does. In this regard, “understand the principle of mathematical induction” means that students should know the procedures of applying the principle, why the principle holds, when the principle becomes invalid and how the principle is used to solve problems.

In the introduction of mathematical induction, the examples adopted should be simple so that students could master and recognise them easily.

In Learning Unit 7 of the Compulsory Part, students understand the formulae for the summation of the arithmetic sequence and that of the geometric sequence. Students may have come across the formulae for the summations of other finite sequences. They may have queries on the correctness of formulae such as $1^2+2^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$. Teachers may guide students to discover the formula as shown below.

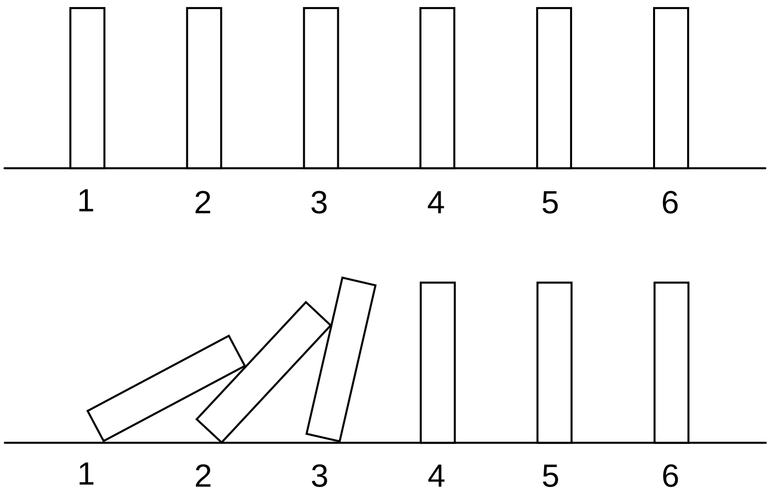
Teachers may ask students to guess the formula for the sum of the first n odd positive integers by considering the following:

$$\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
&\cdots\cdots \\
&\cdots\cdots \\
1+3+5+\ldots+(2n-1) &= ?
\end{aligned}$$

The formula $1+3+5+\ldots+(2n-1) = n^2$ is true for n = 1, 2 and 3. However, how can we make sure that it is true for all positive integers? As a result, it is necessary to prove the statement. Mathematical induction is one of the useful tools.

Teachers may ask students to explore the formulae for the summations of some finite sequences such as $1^2+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ or $1^3+2^3+\ldots+n^3 = \left(\frac{1}{2}n(n+1)\right)^2$ before using the principle of mathematical induction to prove these formulae.

Teachers may explain the principle by referring to a game of dominoes.



The two steps in the principle of mathematical induction are crucial:

(1) Prove that "P(1) is true".

(2) Prove that "If P(k) is true, then P(k+1) is also true (where k is a positive integer)".

Teachers may use the following counter-examples to illustrate that if the two above steps are incomplete, we cannot prove that P(n) is true for all positive integers n.

(a) For any positive integer n, n^2+n+17 is prime.[1]

(b) For any positive integer n, $2^{2^n}+1$ is prime.[2]

---

1 $n^2 + n + 17$ is prime for n =1,2,3,…,15.
When n = 16, $n^2 + n + 17 = 16^2 + 16 + 17 = 17^2$, which is not a prime.

2 Fermat conjectured that all numbers of the form $2^{2^n}+1$ are prime (which are called Fermat numbers) for all positive integers n. He only verified that the proposition holds when n = 1,2,3,4. Later Euler discovered that the 5th Fermat number is not a prime. He showed that $2^{2^5}+1 = 4294967297 = 641\times 6700417$.

(c) For any positive integer n, $1+2+3+\ldots+n=\frac{n(n+1)}{2}+2$.

(a) and (b) are examples of incomplete induction, in which "P(n) is true" for a finite number of cases. In the process of proving the statement, step 2 is incomplete. As a result, the statement " P(n) is true for all positive integers n" cannot be proved.
Example (c) shows that although we can prove that the statement is true for step 2, P(n) is not true because P(1) is false.

Teachers could use more examples to demonstrate how to use the principle of mathematical induction. For examples:

Prove that, for any positive integer n,

(a) $1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\ldots+(n-1)\cdot 2+n\cdot 1=\frac{1}{6}n(n+1)(n+2)$.

(b) 23^n-1 is divisible by 11.

(c) 7^n+3n-1 is divisible by 9.

(d) $a^{2n-1}+b^{2n-1}$ is divisible by $a+b$.

Teachers should remind students to pay attention to the proper use of the following terms in drawing conclusions: numbers, integers, positive numbers and positive integers.

Students should also learn some common variations of the principle. They should be able to prove propositions like "a^n-b^n is divisible by $a+b$ for all positive even integers n" by modifying the 2 steps in using the principle of mathematical induction. However, more complicated variations of the principle of mathematical induction such as the examples below are not required:

(1) P(1) is true.

(2) If P(n) is true for $1\le n\le k$, then P(k+1) is also true (where k is a positive integer).

or

(1) P(1) and P(2) are true.

(2) If P(k–1) and P(k) are true, then P(k+1) is also true.

Proving propositions involving inequalities is not required.

Teachers could apply mathematical induction to demonstrate to students the proof of the binomial theorem in the next Learning Unit.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
3. Binomial Theorem	3.1 expand binomials with positive integral indices using the Binomial Theorem	3

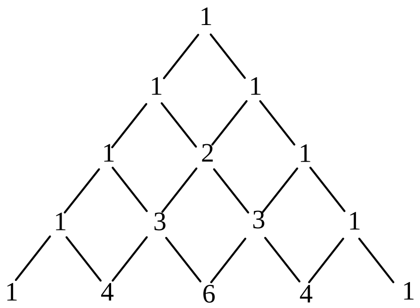
Explanatory Notes:

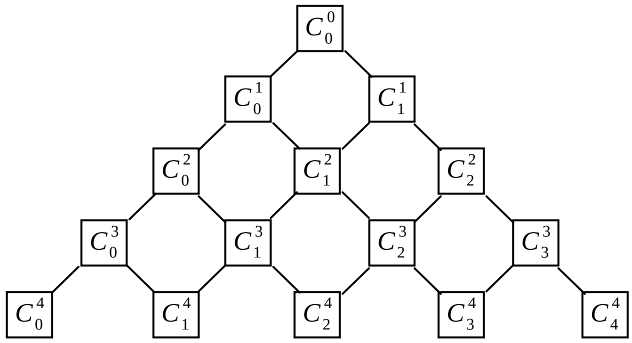
Students should understand how to prove the Binomial Theorem by the principle of mathematical induction.

The definition of C_r^n has been discussed in Learning Unit "Permutation and combination" of the Compulsory Part. Thus, a combinatorial approach can be used to prove the Binomial Theorem. The coefficient of the term a^3b^2 in the expansion of the expression $(a+b)^5$ can be considered as the number of combinations of choosing two b's in the expansion of $(a+b)(a+b)(a+b)(a+b)(a+b)$. It is not difficult for students to understand that C_2^5 is the coefficient of a^3b^2 in the expansion of $(a+b)^5$.

In the introduction, students could be asked to find the coefficient of each term in the expansion $(a + b)^n$ by a combinatorial approach and compare the numerical values with those in Pascal's triangle[3] on the left.

[3] The arrangement of the binomial coefficients in a triangle is named after Blaise Pascal as he included this triangle with many of its application in his treatise, *Traité du triangle arithmétique* (1654). In fact, in the 13th century, Chinese mathematician Yang Hui (楊輝) presented the triangle in his book 《詳解九章算法》(1261) and pointed out that Jia Xian (賈憲) had used the triangle to solve problems. Thus, the triangle is also named Yang Hui's Triangle (楊輝三角) or Jia Xian's Triangle (賈憲三角).





In general, $(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_{n-1}^n ab^{n-1} + C_n^n b^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r$.

To represent the presentation of the binomial expansion in a more concise form, it is natural and necessary to introduce the summation notation ($\sum$) to students. It should be aware that tedious calculations involving the summation notation is not required.

As the Binomial Theorem is a learning unit in the Foundation Knowledge Area, problems and examples involved should be simple and straightforward. In this connection, the following contents are not required:

- expansion of trinomials
- the greatest coefficient, the greatest term and the properties of binomial coefficients
- applications to numerical approximation

The Binomial Theorem can also be used to prove $\frac{d}{dx}(x^n) = nx^{n-1}$ (where n is a positive integer) from first principles.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
4. More about trigonometric functions	4.1 understand the concept of radian measure 4.2 find arc lengths and areas of sectors through radian measure 4.3 understand the functions cosecant, secant and cotangent and their graphs 4.4 understand the identities $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ 4.5 understand compound angle formulae and double angle formulae for the functions sine, cosine and tangent, and product-to-sum and sum-to-product formulae for the functions sine and cosine	11

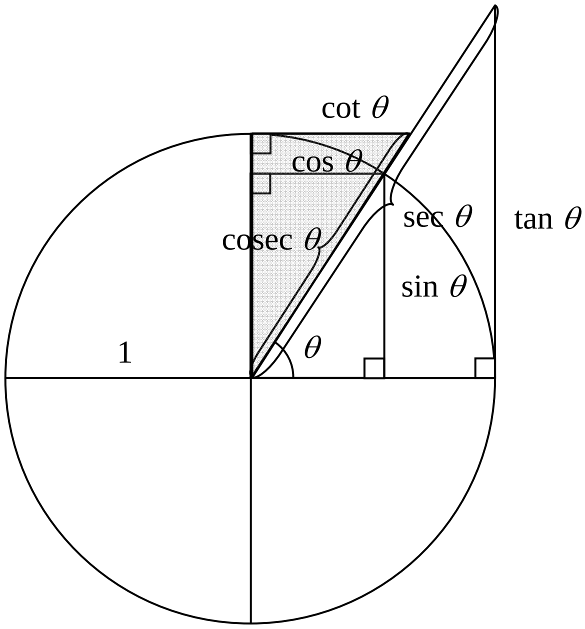
Explanatory Notes:

At KS3, students learnt how to measure angles in degrees and find the arc lengths and areas of sectors by using proportions. In Module 2, students will learn how to measure angles in radians and to find the arc lengths and areas of sectors by formulae in radians. The formula $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ in the Remarks column of Learning Objective 6.2 only holds when θ is in radians. Thus, in dealing with problems in Calculus, all angles related to trigonometric functions are in radians.

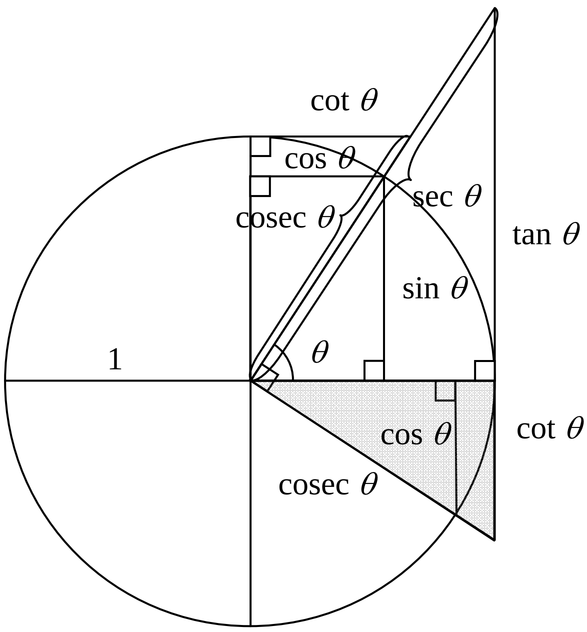
In Learning Objective 13.1 of the Compulsory Part, students have learnt the trigonometric functions sine, cosine and tangent, and their graphs and properties (including maximum and minimum values and periodicity). Students are expected to learn the other three trigonometric functions (cosecant, secant and cotangent), their graphs and properties in this Module. As students have to sketch and compare graphs of various types of functions including trigonometric functions in Learning Objective 9.1 of the Compulsory Part, it is natural to expect students to be able to sketch the graphs and properties of the other three trigonometric functions from those of the three they learnt. In this connection, teachers may start to guide students to sketch the graph of $y = \frac{1}{f(x)}$ from the given graph of $y = f(x)$ and then explore the graph of y = cosecθ from that of $y = \sin\theta$. Students should be able to explore and understand the domains, maximum or minimum values, symmetry and periodicity of the function y = cosec θ. They can further sketch the graphs of secant and cotangent and discover their properties.

Students at KS3 should have learnt the identity $\sin^2\theta + \cos^2\theta = 1$. They should be able to derive from it the other two identities $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. Students are also expected to simplify trigonometric expressions by using these identities.

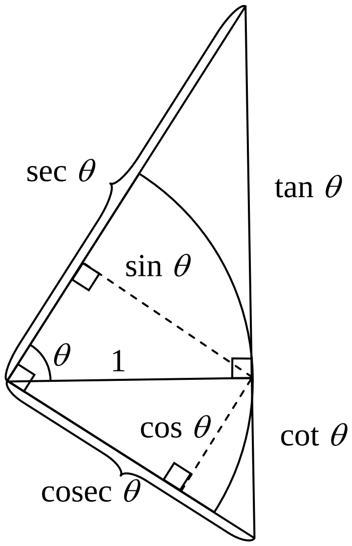
Teachers may use the following figure to illustrate the six trigonometric ratios and their relations and guide students to discover the above identities. The circle in the figure below is a unit circle.



By rotating the shaded triangle of the above figure to a new position, the following figure is obtained.



By adding two dotted lines in the figure, we have:



The trigonometric identities $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ can be observed from the figure above. It is noted that all angles in the proof shown in the above figures are acute.

Students will find the list of formulae in the Remarks column of Learning Unit 4 on pp.68-69 of the C&A Guide very useful in solving problems related to trigonometric functions.

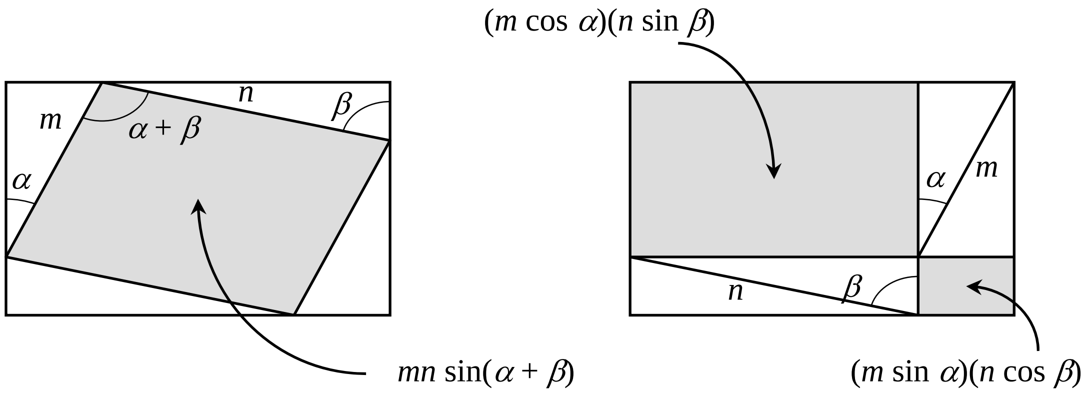
Teaches may use diagrams to derive the compound angle formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \text{ and } \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Other formulae could then be derived from these four basic formulae.

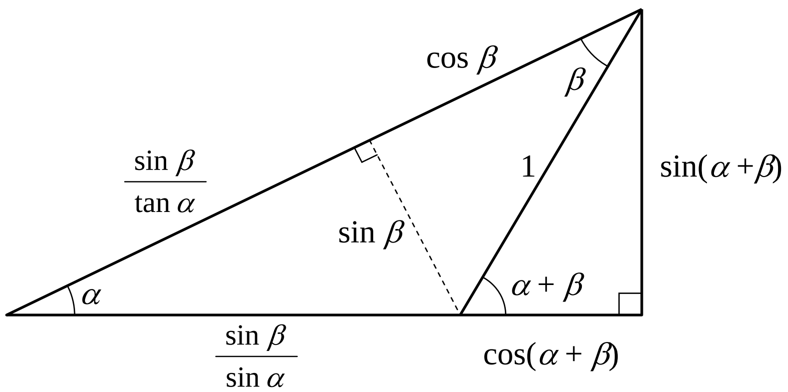
Different related proofs of the above four basic formulae could be found in reference books/textbooks. Some non-traditional proofs are included here for teachers' reference. It is reminded that, in general, there are restrictions on the sizes of angles in the proofs which depend on diagrams.

Example 1 Prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.



The shaded area in the diagram on the left	=	Total shaded area in the diagram on the right
$mn\sin(\alpha+\beta)$	=	$(m\sin\alpha)(n\cos\beta) + (m\cos\alpha)(n\sin\beta)$
$\sin(\alpha+\beta)$	=	$\sin\alpha\cos\beta + \cos\alpha\sin\beta$

Example 2 Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.



$$\sin\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta + \dfrac{\sin\beta}{\tan\alpha}}$$

$$\therefore \qquad \sin(\alpha+\beta) = \sin\alpha\left(\cos\beta + \frac{\sin\beta}{\tan\alpha}\right)$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

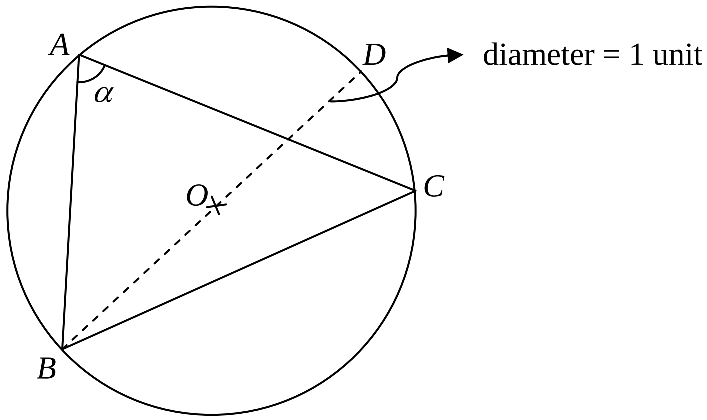
and
$$\cos\alpha = \frac{\cos(\alpha+\beta) + \dfrac{\sin\beta}{\sin\alpha}}{\cos\beta + \dfrac{\sin\beta}{\tan\alpha}}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta + \frac{\cos^2\alpha\sin\beta}{\sin\alpha} - \frac{\sin\beta}{\sin\alpha}$$

$$\therefore \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

<u>Example 3</u> (This example can be considered as an application of Ptolemy's Theorem in Learning Unit 18 of the Compulsory Part)

(a) In the following figure, BOD is a diameter of the circumcircle of ΔABC.



Join CD.

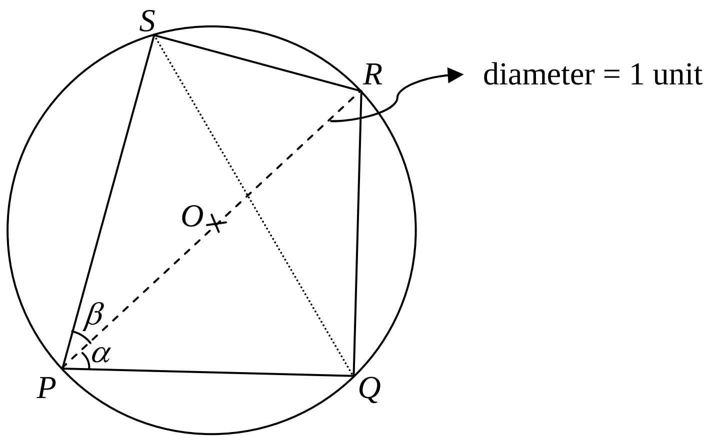
$\angle BDC = \alpha$, $\angle BCD = 90°$.

$\sin\alpha = \dfrac{BC}{1}$

$BC = \sin\alpha$

(b) The following shows the proof of $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ by using Ptolemy's Theorem.

Let $PQRS$ be a cyclic quadrilateral lying on its circumcircle with diameter PR = 1 unit.



As PR is a diameter, $\angle PQR = 90°$, $\angle RSP = 90°$.

$PQ = \cos\alpha$, $QR = \sin\alpha$, $RS = \sin\beta$, $SP = \cos\beta$.

By (a), $QS = \sin(\alpha+\beta)$.

By Ptolemy's Theorem, $PQ\times RS+QR\times SP=PR\times QS$.

$\therefore$ $\cos\alpha\sin\beta + \sin\alpha\cos\beta = 1\times\sin(\alpha+\beta)$.

i.e. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ and $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ can be derived from the double angle formulae. These formulae, together with the product-to-sum and sum-to-product formulae, are important tools in finding integrals. Half angle formulae, triple angle formulae, *t*-formulae and the "subsidiary angle form" are not required in teaching this Learning Unit.

In Learning Objective 13.2 of the Compulsory Part, students should be able to solve simple trigonometric equations with solutions in the interval from 0° to 360°. In this regard, students should be able to solve trigonometric equations with solutions from 0 to 2π only. The discussion of this section can be applied to solve optimisation problems in Learning Objective 8.4.

Learning Unit	Learning Objective	Time
Foundation Knowledge Area		
5. Introduction to the number *e*	5.1 recognise the definitions and notations of the number *e* and the natural logarithm	1.5

Explanatory Notes:

Learning Units 3 and 5 of the Compulsory Part belong to the Non-foundation Topics, in which the exponential and logarithmic functions and their graphs have been discussed. The number *e* and the natural logarithm are introduced as important concepts in mathematics. They are crucial in the learning of Calculus. Students will learn the exponential function e^x in this Learning Unit.

In general, there are two common approaches to introduce *e*:

(1) $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$

(2) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Module 2 (Algebra and Calculus) emphasises the first approach.

The following calculation of the amount at compound interest can be employed as an example to illustrate *e*.

An amount of money is deposited in a bank for one year at an interest rate of 1% p.a. What will be the amount if it is compounded

(i) quarterly?
(ii) monthly?
(iii) daily?
(iv) hourly?
(v) every second?

The discussion of the limit $\lim_{n\to\infty}(1+\frac{1}{n})^n$ can be introduced from these questions.

The Monotone Convergence Theorem[4] is required to prove the existence of *e*. Nevertheless, it is not included in this Curriculum. Therefore, students are not required to prove the existence of the limit.

Furthermore, it can be showed to students that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$.

The second approach involves the expansion of the binomial expression $(1+\frac{x}{n})^n$ and is adopted in Module 1 (Calculus and Statistics).

$$
\begin{aligned}
(1+\frac{x}{n})^n &= \sum_{r=0}^{n} C_r^n (\frac{x}{n})^r \\
&= C_0^n(\frac{x}{n})^0 + C_1^n(\frac{x}{n})^1 + C_2^n(\frac{x}{n})^2 + C_3^n(\frac{x}{n})^3 + \ldots + C_r^n(\frac{x}{n})^r + \ldots + C_n^n(\frac{x}{n})^n \\
&= 1 + \frac{n}{1!}\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\frac{x^r}{n^r} + \ldots \\
&\quad + \frac{n(n-1)\ldots(n-n+1)}{n!}\frac{x^n}{n^n} \\
&= 1 + x + (1-\frac{1}{n})\frac{x^2}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{x^3}{3!} + \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{r-1}{n})\frac{x^r}{r!} + \ldots \\
&\quad + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})\frac{x^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
e^x &= \lim_{n\to\infty}(1+\frac{x}{n})^n \\
&= \lim_{n\to\infty}\Big[1 + x + (1-\frac{1}{n})\frac{x^2}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{x^3}{3!} + \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{r-1}{n})\frac{x^r}{r!} + \\
&\qquad \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})\frac{x^n}{n!}\Big] \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}
$$

By putting $x = 1$, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots$$

and, by using a calculator, it can be shown that the value of *e* converges approximately to 2.71828.

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[4] The Monotone Convergence Theorem states that every monotonic increasing sequence which is bounded above converges and every monotonic decreasing sequence which is bounded below converges.

Teachers should remind students that the natural logarithmic function possesses all properties of the common logarithm function. Teachers need not treat the natural logarithmic function as a new category of functions. The formula for change of base, especially in finding derivatives of logarithms of different bases, is very important in the following learning units. Therefore, the content of Learning Objective 3.3 of the Compulsory Part involving the formula for change of base should be reviewed.

This Learning Unit can also be introduced before teaching Learning Objective 6.1.

Calculus Area

The Calculus Area consists of "Limits and Differentiation" and "Integration".

Students need to master the concepts of functions, their graphs and properties to study "Limits and Differentiation". Besides, the techniques of manipulating surds help them tackle many problems of limits. Students can apply the Binomial Theorem to prove the formula $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a positive integer.

Teachers can further lead students to appreciate the beauty of mathematics such as the irrational number e, the function e^x and its derivative. Students can also discover the importance of differentiation to solve problems related to rate of change, maximum and minimum.

The indefinite integral and differentiation are related as reverse processes to each other. The Fundamental Theorem of Calculus links up the two apparently different concepts. At this stage, one of the applications of the definite integral is to find the area of a plane figure and the volume of a solid of revolution. Students can also appreciate how to apply the definite integral to calculate the areas of non-rectilinear figures such as areas of circles, etc.

Learning Unit	Learning Objective	Time
Calculus Area		
Limits and Differentiation		
6. Limits	6.1 understand the intuitive concept of the limit of a function 6.2 find the limit of a function	5

Explanatory Notes:

"Limit" is the most fundamental concept in Calculus. After studying Learning Units 2 and 9 of the Compulsory Part, students should have a comprehensive understanding of the concepts of various functions and their graphs. However, the functions they met are, in general, continuous. In this Learning Unit, students will encounter a variety of discontinuous functions and discuss some properties of discontinuous functions. The introduction of this topic can be started from the graphs of continuous functions as well as from those of discontinuous functions. The intuitive concept of the limit of a function can also be developed. Graphing software is very useful in the exploration of the graphs of functions. It should be noted that the rigorous definition of the limit of a function is not required in this Curriculum.

To illustrate the continuity of functions, the absolute value function $|x|$, the signum function sgn(x), the ceiling function $\lceil x \rceil$ and the floor function $\lfloor x \rfloor$ may be introduced to students. It should be noted that these functions are examples only. The formal definition of continuity is not required. Manipulations of the absolute value function, the differentiation and integration of the absolute value function are not required.

The following examples can be employed to discuss with students whether certain limits exist by considering the graphs of the corresponding functions.

(a) Let $f(x) = |x|$, find $\lim\limits_{x \to 0} f(x)$.

(b) Let $f(x) = \begin{cases} x^2 , & x \geq 1 \\ x - 1 , & x < 1 \end{cases}$, find $\lim\limits_{x \to 1} f(x)$.

(c) Let $f(x)=\dfrac{2}{x-1}$, find $\lim\limits_{x\to 1} f(x)$.

For more able students, teachers may ask them to sketch the graph of $h(x)=f(x)+g(x)$, where $f(x)=\begin{cases} x+1, & x\ge 1 \\ 2x, & x<1 \end{cases}$ and $g(x)=\begin{cases} -x, & x\ge 0 \\ x+2, & x<0 \end{cases}$, and find the value(s) of x where the graph is discontinuous.

The formal proofs of the theorems on the limits of the sum, difference, product, quotient, and scalar multiple of functions, and those on the limits of composite functions are not required, but the conditions that these theorems hold should be stated clearly. For example, it is important to state clearly that $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)$ if both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Besides, students may be asked to construct examples such that $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)$ do not hold.

Students should know how to find the limits of simple functions, such as $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+3}-\sqrt{3}}$. The rationalisation of denominators in Learning Unit 1 is an important tool in dealing with this kind of problems. Students should also apply a similar technique to rationalise, if necessary, the numerator to find the limits of functions.

Students should understand two important formulae in limits, namely $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ (θ is in radians) and $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

The formula $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ = 1 is crucial for finding the derivatives of trigonometric functions. In proving this formula, it should be noted that it may involve the application of Sandwich Theorem. An easy way to deal with this limit is to use a diagram to compare the areas of two triangles and a sector.

To introduce the formula $\lim_{x\to 0}\frac{e^x-1}{x}=1$, the substitution x = ln(1+h) and the definition of e^x can be used to prove the formula. This formula can be used to find the derivative of e^x. Students may be asked to guess the value of the limit $\frac{e^x-1}{x}$ as x approaches 0 by referring to the formula $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$.

Finding the limit of a rational function at infinity is required except those involving the application of the technique of partial fractions.

Learning Unit	Learning Objective	Time
Calculus Area		
Limits and Differentiation		
7. Differentiation	7.1 understand the concept of the derivative of a function 7.2 understand the addition rule, product rule, quotient rule and chain rule of differentiation 7.3 find the derivatives of functions involving algebraic functions, trigonometric functions, exponential functions and logarithmic functions 7.4 find derivatives by implicit differentiation 7.5 find the second derivative of an explicit function	14

Explanatory Notes:

Students should be able to find, from first principles, the derivatives of elementary functions, such as constant functions, x^n (n is a positive integer), $\sqrt{x}$, $\sin x$, $\cos x$, e^x and $\ln x$. They are expected to apply the technique of rationalisation to find the derivatives of functions, such as $\frac{1}{\sqrt{x+1}}$, from first principles. The Binomial Theorem is required to support the proof of $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a positive integer, from first principles. Teachers may also prove this formula by mathematical induction.

Students should be familiar with notations such as y', $f'(x)$ and $\frac{dy}{dx}$ for derivatives. Tests of differentiability of functions are not required.

Students should be able to apply the addition rule, product rule, quotient rule and chain rule to find the derivatives of functions. It should be noted that students do not have to learn the concept of composite function in the Compulsory Part. Presentations like $\frac{d}{dx}(\sin^2 x) = \frac{d(\sin^2 x)}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} = 2\sin x \cos x$ would be helpful for students to understand the chain rule.

To find the derivatives of logarithmic functions with base not equal to e, it is needed to use the "change of base" formula learnt in Learning Objective 3.3 (Non-foundation Topics) of the Compulsory Part.

For example, $y = \log_2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{x\ln 2}.$$

It should be explained to students that implicit differentiation is a useful tool in finding derivatives without having to express the dependent variable in terms of the independent variable. Equations like $x^3 - 3xy + y^3 = 3$ and $x = y + y^2$ are examples for illustrating the use of implicit differentiation to find $\frac{dy}{dx}$. It is not easy and sometimes not possible to express y in terms of x for some equations. For the purpose of differentiating, it is not necessary to express y in terms of x.

Equations such as $y = (x^2+2)(3x-2)^2(4x+5)^6$ and $y = \left(\frac{2x+1}{2x-1}\right)^4$ can be used as examples to demonstrate the technique of logarithmic differentiation.

Finding the second derivatives of implicit functions has no wide applications in this Module. Students are expected to find the second derivatives of explicit functions only. The second derivatives are useful in finding the extrema of functions in Learning Objective 8.2. Notations such as y'', $f''(x)$ and $\frac{d^2y}{dx^2}$ should be introduced. Third and higher order derivatives are not required.

Learning Unit	Learning Objective	Time
Calculus Area		
Limits and Differentiation		
8. Applications of differentiation	8.1 find the equations of tangents and normals to a curve 8.2 find maxima and minima 8.3 sketch curves of polynomial functions and rational functions 8.4 solve the problems relating to rate of change, maximum and minimum	14

Explanatory Notes:

The contents of Learning Objectives 2.3 and 2.4 (Non-foundation Topics) of the Compulsory Part include the use of the graphical method or the algebraic method to find the maximum/minimum value of a quadratic function. In Module 2 (Algebra and Calculus), differentiation is not restricted to solve optimisation problems of quadratic functions.

In Learning Objective 8.1, students are required to find not only the equations of tangents and normals passing through a given point on a given curve, but also the equations of the tangents passing through an external point. As it involves using the algebraic method to solve simultaneous equations in two unknowns, one linear and one quadratic, students should have a more comprehensive understanding of Learning Objective 5.2 (Non-foundation Topics) of the Compulsory Part.

The first derivative test and the second derivative test may be used to find local extrema (i.e. maxima and minima) of functions. In addition to the local extrema, the values of the function at the endpoints of a closed interval should be considered to determine the global extrema. When $f''(x_0)=0$, the second derivative test is not applicable to determine the extrema at $x=x_0$. In this case, students have to use the first derivative test.

Students need to be able to use the second derivative to determine the concavity and convexity of a function and use these properties to find the points of inflexion of functions.

Teachers should note that curve sketching is restricted to polynomial functions and rational functions only. Certain features of a curve can be observed, or easily obtained, from the equation of the curve, for example,

- symmetry of the curve
- limitations on the values of x and y
- intercepts with the axes
- maximum and minimum points
- points of inflexion
- vertical, horizontal and oblique asymptotes to the curve

It is not necessary to consider all of these features when examining a particular curve. As different features will be considered in different problems, it is required to demonstrate the application of each by different examples.

The properties of convexity and concavity and the concepts of increasing and decreasing functions are useful in curve sketching. The tangent to a curve at a point of inflexion crosses the curve. It may be horizontal, oblique and even vertical. Discussions on finding the oblique asymptote should not involve complicated calculations of limits. It is sufficient for students to deduce the equation of the oblique asymptote to the graph of a rational function by long division. Therefore, students' concepts on the division of polynomials learnt in Learning Objective 4.1 of the Compulsory Part should be consolidated.

Before solving word problems of maxima and minima, students should note the following points:

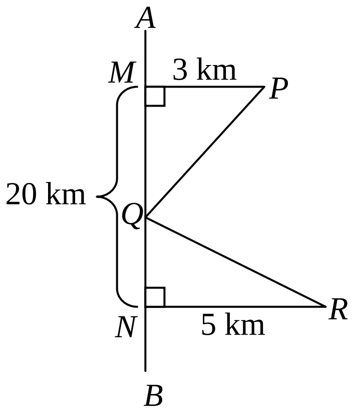
(1) Local maximum and minimum values of a continuous function occur alternately.

(2) If a function has only one turning point, it is obvious, from the nature of the problem, to determine whether it is a maximum or a minimum point.

In solving problems related to extrema, it should pointed out that using derivatives is sometimes not the only way to find the maximum or minimum values of a function. Completing the square is a useful algebraic method to find the maximum or minimum values of quadratic functions. In dealing with word problems, considerations of the physical situations of the problems sometimes provide excellent clues. The following examples can be solved by differentiation. However, the solutions from considering their physical situations are more elegant than those by using the method of Calculus.

Example 1

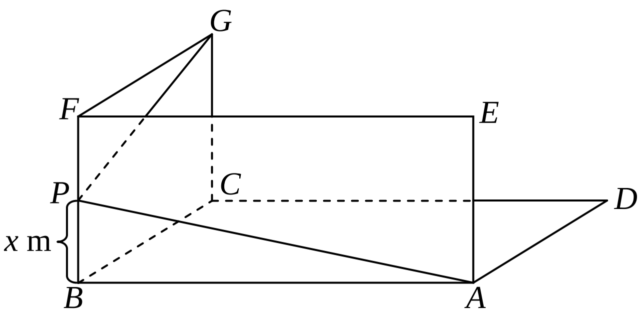
In the figure, P and R are two points on the same side of the river AB. It is intended to walk from the point P to a point Q on the riverside and then to the point R. Where is Q so that the total distance from P to R through Q is the shortest?



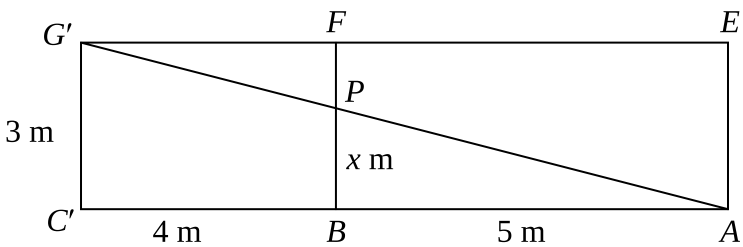
(Hint: Reflect R in AB to get its image R'. The length of the route PQR = PQ+QR = PQ+QR'. As a result, the route PQR is shortest when PQR' is a straight line. By considering similar triangles, MQ = 7.5 km)

Example 2

From the corner A of the floor $ABCD$ of a room, an electric wire is to be laid along the walls $ABFE$ and $EGCB$. Suppose the wire cuts the line FB at the point P where AB = 5 m, BC = 4 m and BF = 3 m. Find x such that the total length L of the wire is a minimum.



(Hint: Imagine that BF is the axis of rotation for $BCGF$ to turn through 90° to be coplanar with $ABFE$. Let G' be the image of G. L is minimum when APG' is a straight line. By considering similar triangles, $x = \frac{5}{3}$)



The problems of maximising profits and average profits in Economics can be used as examples for the application of differentiation in other disciplines.

Learning Unit	Learning Objective	Time
Calculus Area		
Integration		
9. Indefinite integration	9.1 recognise the concept of indefinite integration 9.2 understand the properties of indefinite integrals and use the integration formulae of algebraic functions, trigonometric functions and exponential functions to find indefinite integrals 9.3 understand the applications of indefinite integrals in real-life or mathematical contexts 9.4 use integration by substitution to find indefinite integrals 9.5 use trigonometric substitutions to find the indefinite integrals involving $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{a^2+x^2}$ 9.6 use integration by parts to find indefinite integrals	16

Explanatory Notes:

Students should understand that indefinite integration is the reverse process of differentiation. If $\frac{d}{dx}F(x) = f(x)$, then $\int f(x)\,dx = F(x)+C$, where C is called the constant of integration. The expression $\int f(x)\,dx$ is called the indefinite integral of $f(x)$. It should be pointed out that indefinite integral is not unique. If $F(x)$ is an indefinite integral of $f(x)$, then $F(x)+C$ (C is a constant) is another. In addition, it should be demonstrated with examples that different methods of integration may lead to answers which look different.

For example, $\int (x+1)^2\,dx = \int (x^2+2x+1)\,dx = \frac{1}{3}x^3 + x^2 + x + C_1$ and

$\int (x+1)^2\,dx = \int (x+1)^2\,d(x+1) = \frac{1}{3}(x+1)^3 + C_2$.

Students may be asked to show that $C_1 = C_2 + \frac{1}{3}$ and note that these two answers only differ by a constant term.

The formulae in the Remarks column of Learning Objective 9.2 should be understood instead of rote memorization. In deducing the formula $\int \frac{1}{x}\, dx = \ln|x| + C$, the absolute value $|x|$ should be introduced. Students are not required to learn the indefinite integral $\int f(x)\,dx$, where $f(x)$ involves absolute values.

Integration by substitutions and integration by parts are useful tools to find indefinite integrals.

In trigonometric substitutions, functions such as $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ may appear in the answers. As students do not have the concepts of inverse functions, teachers should discuss with students the notations of these inverses and introduce their principal values. It should be noted that integrands containing inverse trigonometric functions are not required.

Problems involving partial fractions are not included in this Module.

It is appropriate that the use of integration by parts is restricted to at most two times in finding an integral to avoid tedious calculations. Reduction formulae of integration are not included.

Learning Unit	Learning Objective	Time
Calculus Area		
Integration		
10. Definite integration	10.1 recognise the concept of definite integration 10.2 understand the properties of definite integrals 10.3 find definite integrals of algebraic functions, trigonometric functions and exponential functions 10.4 use integration by substitution to find definite integrals 10.5 use integration by parts to find definite integrals 10.6 understand the properties of the definite integrals of even, odd and periodic functions	11

Explanatory Notes:

The basic definition of the definite integral as the limit of a sum should be introduced to students. Students may confuse the notation of the definite integral with that of the indefinite integral. It is expected to introduce to students the Fundamental Theorem of Calculus as a connection between the two concepts. The proof of the Fundamental Theorem of Calculus is also required.

Students should understand the concept of dummy variables in definite integrals. All properties of definite integrals in the Remarks column of Learning Objective 10.2 should be highlighted to students.

Discussions on properties of the definite integrals for even, odd and periodic functions help students deepen their understanding on definite integrals.

Students should understand the application of integration by substitution in proving the properties stated in the Remarks column of Learning Objective 10.6.

Teachers should note that the followings are not required:

- the evaluation of the definite integral $\int_a^b f(x)\,dx$, where $f(x)$ involves absolute values
- reduction formulae

- the evaluation of the sum to infinity of a sequence by using a definite integral
- improper integrals
- the inequality $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Learning Unit	Learning Objective	Time
Calculus Area		
Integration		
11. Applications of definite integration	11.1 understand the application of definite integrals in finding the area of a plane figure 11.2 understand the application of definite integrals in finding the volume of a solid of revolution about a coordinate axis or a line parallel to a coordinate axis	7

Explanatory Notes:

In this Module, the applications of definite integration only confine to the calculations of areas of plane figures and volumes of solids of revolution. A geometric demonstration on the relationship between the definition of the definite integral and the area of a plane figure may be given. Students may be led to appreciate the application of definite integration in providing a rigorous way to prove the formulae of the area of circle, the volume of right circular cone and the volume of sphere. The difference between the volumes of solids of revolution about the coordinate axes or straight lines parallel to them should be discussed.

Both the "disc method" and the "shell method" are included. An intuitive geometric explanation should be given though a formal proof is not required. In some cases, the "shell method" is better than the "disc method" in finding the volume of revolution of certain objects. For example, in finding the revolution of the area bounded by the curve $y = \sin x + x$ and the lines $y = 0$, $x = \frac{\pi}{2}$ about the y-axis, it is easy to use the "shell method". Teachers may compare the two methods in finding the volume of the same solid of revolution.

Finding the volume of a hollow solid is required.

Algebra Area

The Algebra Area consists of "Matrices and Systems of Linear Equations" and "Vectors".

In this Area, students encounter a new mathematics structure, "Matrices" , in Algebra. They will find that the multiplication of matrices is not commutative. This new concept is different from what they have experienced in the past. They are required to understand the concepts, operations and properties of matrices, the existence of inverse matrices and the determinants. Determinants are important tools to investigate the properties of matrices.

At KS3, students solved linear equations in two unknowns by the algebraic method and the graphical method. Those who studied the enrichment topic "explore simultaneous equations that are inconsistent or that have no unique solution" at KS3 may have the preliminary concepts of "consistency" and "inconsistency". In this Area, students further explore the conditions of consistency or inconsistency in a system of linear equations. They should be able to use Cramer's rule, inverse matrices and Gaussian elimination to solve systems of linear equations. They should also understand the strengths and weaknesses of each method and how to choose appropriate methods to solve problems.

In order to extend students' knowledge in Algebra Area, the concepts, operations and properties of vectors should be introduced. The scalar product and the vector product are two useful tools to investigate the geometric properties of vectors including parallelism and orthogonality. In addition, students can learn to use the vector method to find the volume of a parallelepiped, the angle between two vectors and the area of a triangle or a parallelogram, etc.

Learning Unit	Learning Objective	Time
Algebra Area		
Matrices and Systems of Linear Equations		
12. Determinants	12.1 recognise the concept and properties of determinants of order 2 and order 3	3

Explanatory Notes:

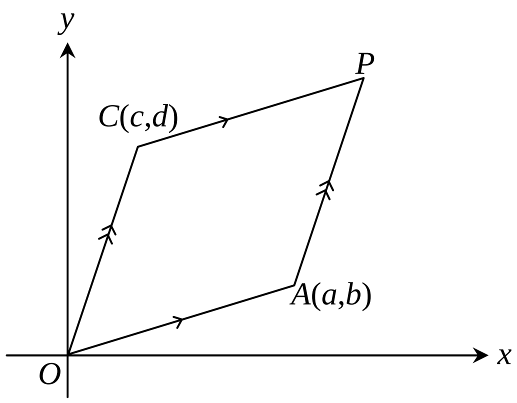
Determinant is a vital tool in the learning of matrices and systems of linear equations.

Students are expected to learn the basic operations and basic properties of determinants of order 2 and order 3. The properties are shown in the Remarks column on pp.80-81 of the C&A Guide.

Students should know that both $|A|$ and $\det(A)$ are common notations of the determinant of the matrix A.

Teachers could introduce one of the uses of determinant as stated below.

In the figure, $OAPC$ is a parallelogram passing through the origin O where $A=(a,b)$ and $C=(c,d)$.



Area of parallelogram $OAPC$ = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Learning Unit	Learning Objective	Time
Algebra Area		
Matrices and Systems of Linear Equations		
13. Matrices	13.1 understand the concept, operations and properties of matrices 13.2 understand the concept, operations and properties of inverses of square matrices of order 2 and order 3	9

Explanatory Notes:

The general proof of $|AB| = |A||B|$ in Learning Objective 13.1 is a bit difficult for average students. In teaching this property, it can be verified by examples without proving. However, teachers may have more in-depth discussions on this property with students for square matrices up to order 3 since the proofs are simpler.

Although the identity matrix and zero matrix are not listed in this C&A Guide, their definitions and properties should be discussed. These two special matrices are important in the introduction of the multiplicative inverse and the additive inverse. The non-commutative property of matrix multiplication, i.e. $AB \neq BA$, should be emphasised as it contradicts to students' past experience.

In finding the inverse of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, students need to solve the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for unknowns x, y, z and w. To check whether the inverse matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ exists, it is natural to consider the value of $ad - bc$ which is the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. The concept and properties of determinant in Learning Unit 12 and the contents of this Learning Unit are closely related. In teaching these topics, they can be absorbed into each other.

Students are expected to understand the concepts, operations and properties of inverse square matrices of order 2 and order 3. Students need to determine whether a matrix is invertible and to find the inverse of an invertible matrix, such as using the adjoint matrix, using elementary row operations, etc. In addition, in some circumstances, students may need to use the principle of mathematical induction to prove propositions involving matrices.

Learning Unit	Learning Objective	Time
Algebra Area		
Matrices and Systems of Linear Equations		
14. Systems of linear equations	14.1 solve the systems of linear equations of order 2 and order 3 by Cramer's rule, inverse matrices and Gaussian elimination	6

Explanatory Notes:

At KS3, the algebraic and graphical methods in solving linear equations in two unknowns were discussed. The enrichment topic "explore simultaneous equations that are inconsistent or that have no unique solution", "consistency" and "inconsistency" of systems of linear equations were also introduced. In this Learning Unit, the methods of solving systems of linear equations of order 2 and order 3 by Cramer's rule, inverse matrices and Gaussian elimination are further explored. The systems of linear equations involved may be either homogeneous or non-homogeneous. At this stage, the meaning of "consistency" and "inconsistency" should be clearly introduced to students.

Cramer's rule is an important topic of determinants. By Cramer's rule, it is known that for the system of linear equations $A\,\mathbf{x} = \mathbf{b}$, if Δ is the determinant of the coefficient matrix and $\Delta \neq 0$, the system has a unique solution. If $\Delta = 0$, Cramer's rule cannot be used. Teachers may discuss with students a more general result:

$$\Delta \cdot x = \Delta_x , \quad \Delta \cdot y = \Delta_y \quad \text{and} \quad \Delta \cdot z = \Delta_z \ (*)$$

Δ_x is the determinant obtained by replacing the first column of the coefficient matrix by the column vector **b**, Δ_y is the determinant obtained by replacing the second column of the coefficient matrix by the column vector **b** and Δ_z is the determinant obtained by replacing the third column of the coefficient matrix by the column vector **b**. The above results still hold even when Δ is zero. It can further be deduced from (*) to have the following conclusions.

Case	Condition	Conclusion
1	$\Delta \neq 0$	The system has a unique solution.
2	$\Delta = 0$ and at least one of Δ_x , Δ_y or $\Delta_z \neq 0$	The system has no solutions.
3	$\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$	The system has no solutions or infinitely many solutions.

In Case 1, the system has a unique solution and $x=\dfrac{\Delta_x}{\Delta}, y=\dfrac{\Delta_y}{\Delta}, z=\dfrac{\Delta_z}{\Delta}$.

In Case 2, as the given condition contradicts (*), the system has no solutions.

In Case 3, teachers may use the following examples to illustrate that the systems have no solutions or infinitely many solutions.

$$\begin{cases} x+y+z=1 \\ x+y+z=2 \\ x+y+z=3 \end{cases} \quad \text{(no solutions)} \qquad \begin{cases} x+y+z=3 \\ 2x+2y+2z=6 \\ 3x+3y+3z=9 \end{cases} \quad \text{(infinitely many solutions)}$$

Students may wonder why the system does not have a unique solution in Case 3. In Case 3, the system may or may not have solutions. Suppose that the system $A\,\mathbf{x} = \mathbf{b}$ has a solution, say $\mathbf{x}_1$, i.e. $A\,\mathbf{x}_1=\mathbf{b}$. As $\Delta = 0$ for Case 3, by applying the theorem stated in the Remarks column on pp.83 of the C&A Guide, the system of homogeneous equations $A\,\mathbf{x} = \mathbf{0}$ must have at least one non-zero solution, say $\mathbf{x}_2$, i.e. $A\mathbf{x}_2 = \mathbf{0}$. As $A(\mathbf{x}_1 + \lambda\,\mathbf{x}_2) = A\mathbf{x}_1 + \lambda A\mathbf{x}_2 = \mathbf{b} + \lambda \cdot \mathbf{0} = \mathbf{b}$, where λ is any real number, then $\mathbf{x}_1 + \lambda\,\mathbf{x}_2$ is a set of solutions to the systems. Therefore, if the system has a solution, it must have infinitely many solutions.

Students should understand the proof of the theorem "a system of homogeneous linear equations in three unknowns has nontrivial solutions if and only if the coefficient matrix is singular". Teachers can use a system of homogeneous linear equations in two unknowns to discuss this theorem with students, and introduce the meaning of "necessary and sufficient conditions". Student should also understand that a system of homogeneous linear equations is always consistent and know the way to find its nontrivial solution.

Students should be able to solve systems of linear equations by Gaussian elimination in addition to the use of Cramer's rule. By setting up the augmented matrix, elementary row operations can be applied to solve systems of linear equations.

Matrix is another important tool for solving systems of linear equations. Rewriting a system of linear equations in matrix form, if the inverse of the coefficient matrix exists, the system can be solved by using the inverse matrix. Students should recognise that this method becomes invalid if the inverse matrix does not exist. Teachers may demonstrate the linkage between matrices, determinants and elementary row operations in solving systems of linear equations.

Learning Unit	Learning Objective	Time
Algebra Area		
Vectors		
15. Introduction to vectors	15.1 understand the concepts of vectors and scalars 15.2 understand the operations and properties of vectors 15.3 understand the representation of a vector in the rectangular coordinate system	5

Explanatory Notes:

One of the objective of Learning Unit "More about 3-D Figures" at KS3 is to develop students' spatial sense. Students tried to investigate the properties of straight lines through an analytic approach in Learning Unit "Coordinate Geometry of Straight Lines".

All vectors are restricted to $\mathbf{R}^2$ or $\mathbf{R}^3$ in the discussion of the vector properties. The magnitude and direction are two key concepts of vectors. At the same time, it should be emphasised that the concepts of vectors are different from those of the straight lines they studied at KS3. Students should understand the formulae $|\overrightarrow{OP}|=\sqrt{x^2+y^2}$ in $\mathbf{R}^2$ and $|\overrightarrow{OP}|=\sqrt{x^2+y^2+z^2}$ in $\mathbf{R}^3$, the formulae $\sin\theta=\dfrac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta=\dfrac{x}{\sqrt{x^2+y^2}}$ in $\mathbf{R}^2$, and the angle that a non-zero vector makes with the x-axis. It should be stressed that the concept of direction cosines is not required in $\mathbf{R}^3$. Equations of straight lines and planes in $\mathbf{R}^3$ are beyond the scope of this Module.

For the operations and properties of vectors, the eight properties in the Remarks column of Learning Objective 15.2 should be discussed.

Learning Unit	Learning Objective		Time
Algebra Area			
Vectors			
16. Scalar product and vector product	16.1	understand the definition and properties of the scalar product (dot product) of vectors	5
	16.2	understand the definition and properties of the vector product (cross product) of vectors in $\mathbf{R}^3$	

Explanatory Notes:

Students should be able to distinguish between the scalar product and the vector product. Geometric meanings and properties of the scalar product and the vector product should further be discussed. It should be noted that, in Learning Objective 16.2, all vectors concerned should be in $\mathbf{R}^3$.

Each of the following definitions can be adopted as a starting point to introduce the vector product:

(1) For any non-zero and non-parallel vectors **a** and **b** in $\mathbf{R}^3$,

$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where θ is the angle between **a** and **b** (0°≤θ≤180°), $\hat{\mathbf{n}}$ is the unit vector orthogonal to both **a** and **b**, and **a**, **b**, **a**×**b** follow the right-hand rule.

Otherwise, $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

(2) For vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$,

$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$.

Alternatively, teachers could also introduce the determinant form of the vector product to students.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

The scalar triple product and its properties should be introduced to students. The determinant form of scalar triple product

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and

$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ should also be introduced.

The properties of determinants can be used to prove the two properties of scalar triple products ($\mathbf{a} \times \mathbf{b}$) · $\mathbf{c}$ = $\mathbf{a}$ · ($\mathbf{b} \times \mathbf{c}$) and ($\mathbf{a} \times \mathbf{b}$) · $\mathbf{c}$ = ($\mathbf{b} \times \mathbf{c}$) · $\mathbf{a}$ = ($\mathbf{c} \times \mathbf{a}$) · $\mathbf{b}$. The geometric meaning of $\mathbf{a}$ · ($\mathbf{b} \times \mathbf{c}$) can be regarded as "the base area of a face of the parallelepiped times its corresponding height". In other words, the scalar triple product is the volume of a parallelepiped.

Learning Unit	Learning Objective	Time
Algebra Area		
Vectors		
17. Applications of vectors	17.1 understand the applications of vectors	8

Explanatory Notes:

At KS3, the conditions for two lines to be parallel or perpendicular were discussed. In this Learning Unit, students should be able to use the properties of vectors to manipulate the concepts of parallelism and orthogonality. A zero vector product of two non-zero vectors gives rise to parallelism. If one non-zero vector is a scalar multiplication of another non-zero vector, the two vectors are parallel. A zero scalar product of two non-zero vectors gives rise to orthogonality. Students can apply concepts of vectors to investigate the division of a line segment and the projection of a vector onto another vector. In addition, the volume of a parallelepiped, the angle between two vectors and the area of a parallelogram can be found by means of scalar triple product, scalar product and vector product respectively.

Teachers should be aware that three dimensional geometry involving the equations of straight lines and equations of planes are not required.

Learning Unit	Learning Objective	Time
Further Learning Unit		
18. Inquiry and investigation	Through various learning activities, discover and construct knowledge, further improve the ability to inquire, communicate, reason and conceptualise mathematical concepts	10

Explanatory Notes:

This Learning Unit aims at providing students with more opportunities to engage in the activities that avail themselves of discovering and constructing knowledge, further improving their abilities to inquire, communicate, reason and conceptualise mathematical concepts when studying other Learning Units. In other words, this is not an independent and isolated Learning Unit and the activities may be conducted in different stages of a lesson, such as motivation, development, consolidation or assessment.

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